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TRACE ON THE BOUNDARY FOR SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. Let L be a second order elliptic differential operator in $\mathbb{R}^d$ with no zero order terms and let E be a bounded domain in $\mathbb{R}^d$ with smooth boundary ∂E. We say that a function h is L-harmonic if $Lh = 0$ in E. Every positive L-harmonic function has a unique representation

$$h(x) = \int_{\partial E} k(x, y)\nu(dy),$$

where k is the Poisson kernel for L and ν is a finite measure on ∂E. We call ν the *trace of* h *on* ∂E.

Our objective is to investigate positive solutions of a nonlinear equation

$$Lu = u^\alpha \quad \text{in } E$$

for $1 < \alpha \le 2$ [the restriction $\alpha \le 2$ is imposed because our main tool is the α-superdiffusion which is not defined for $\alpha > 2$]. We associate with every solution u a pair (Γ, ν), where Γ is a closed subset of ∂E and ν is a Radon measure on $O = \partial E \setminus \Gamma$. We call (Γ, ν) the *trace of* u *on* ∂E. Γ is empty if and only if u is dominated by an L-harmonic function. We call such solutions moderate. A moderate solution is determined uniquely by its trace. In general, many solutions can have the same trace. We establish necessary and sufficient conditions for a pair (Γ, ν) to be a trace, and we give a probabilistic formula for the maximal solution with a given trace.

1. Introduction

1.1. Diffusions. We start from an elliptic differential operator

$$Lu = \sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_i b_i \frac{\partial u}{\partial x^i} \tag{1.1}$$

in $\mathbb{R}^d$. An L-diffusion in $\mathbb{R}^d$ is a Markov process $\xi = (\xi_t, \Pi_x)$ with continuous paths and with the transition function $p_t(x, y)dy$, where $p_t(x, y)$ satisfies the following conditions:

1.1.A. For all t, x,

$$\int_E p_t(x, y)dy = 1.$$

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1.1.B. For all $s, t > 0$ and all x, z,

$$\int_{\mathbb{R}^d} p_s(x,y)dyp_t(y,z) = p_{s+t}(x,z).$$

1.1.C. For every bounded positive continuous function φ,

$$v_t(x) = \int_E p_t(x,y)\varphi(y)dy$$

is the minimal solution of the boundary value problem

$$\begin{aligned} \frac{\partial v}{\partial t} &= Lv \quad \text{for } t > 0, \\ v &\to \varphi \quad \text{as } t \downarrow 0. \end{aligned}$$

The existence of such process is proved, under broad conditions on the coefficients of L, for instance, in [1].

An L-diffusion in an arbitrary open set D is obtained by killing ξ at the first exit time τ from D. Its transition density is given by the formula[1]

$$p_{D,t}(x,y) = p_t(x,y) - \Pi_x\{\tau < t, p_{t-\tau}(\xi_\tau, y)\}.$$

The corresponding Green's function

$$g_D(x,y) = \int_0^\infty p_{D,t}(x,y)dt$$

is finite for all $x \neq y$ if D is bounded. The Green's operator acts on positive Borel functions by the formula

$$G_D\varphi(x) = \int_E g_D(x,y)\varphi(y)m(dy). \tag{1.2}$$

We fix a bounded domain E with boundary ∂E of class $C^{2,\lambda}$, and we drop the subscripts E in notation $p_E, g_E, \dots$. We denote the first exit time from E by ζ.

The name *L-harmonic functions* is used for solutions of the equation $Lh = 0$ in E. Every positive L-harmonic function h has a unique representation

$$h(x) = \int_{\partial E} k(x,y)\nu(dy), \tag{1.3}$$

where k is the Poisson kernel for L (equal to the normal derivative of $g(x,y)$ with respect to y) and ν is a finite measure on ∂E. We call ν the *trace of h on ∂E*.

The following bounds for k can be found in [19, Lemma 1]:

$$c|x-y|^{-d} \le \frac{k(x,y)}{d(x,\partial E)} \le c^{-1}|x-y|^{-d} \qquad \text{for all } x \in E, y \in \partial E \tag{1.4}$$

(the constant $c > 0$ depends only on L and E). Formula (1.4) implies that

$$h(x) \to 0 \quad \text{as } x \to a, x \in E, \tag{1.5}$$

if ν does not charge a neighborhood of a.

Let σ be the measure on ∂E which appears in the representation (1.3) of the L-harmonic function $h = 1$. Then for all x and Γ,

$$\Pi_x\{\xi_\zeta \in \Gamma\} = \int_\Gamma k(x,y)\sigma(dy). \tag{1.6}$$

[1] Writing $\Pi\{\Omega', Y\}$ means $\int_{\Omega'} Y d\Pi$.

1.2. Superdiffusions. A superdiffusion is a mathematical model of a random cloud. The spatial motion of its infinitesimal parts is described by the L-diffusion $\xi = (\xi_t, \Pi_x)$ in E, and the branching mechanism is determined by a parameter $\alpha \in (1, 2]$. To every open subset D of E and to every $\mu \in \mathcal{M}(E)$[2] there corresponds a random measure (X_D, P_μ) on $E \setminus D$, called the *exit measure from* D. X_D describes the mass distribution of the cloud instantaneously frozen on $E \setminus D$, and P_μ is a probability measure corresponding to initial mass distribution μ. All P_μ have the same domain $\mathcal{F}$. For every positive Borel function f,

$$P_\mu \exp\langle -f, X_D\rangle = \exp\langle -u, \mu\rangle, \tag{1.7}$$

where

$$u + \mathcal{E}_D(u) = K_D f \tag{1.8}$$

with

$$\mathcal{E}_D(u)(x) = G_D(u^\alpha)(x), \tag{1.9}$$

$$K_D f(x) = \Pi_x f(\xi_\tau) \tag{1.10}$$

(τ is the first exit time of ξ from D).

We denote by P_x the measure P_{δ_x} corresponding to Dirac's measure at point x.

The joint probability distribution of $X_{D_1}, \dots, X_{D_n}$ is determined by (1.7) and by the Markov property: for every positive $\mathcal{F}_{\supset D}$-measurable Y,

$$P_\mu\{Y \big| \mathcal{F}_{\subset D}\} = P_{X_D} Y, \tag{1.11}$$

where $\mathcal{F}_{\subset D}$ is the σ-algebra generated by $X_{D'}$ with $D' \subset D$ and $\mathcal{F}_{\supset D}$ is the σ-algebra generated by $X_{D''}$ with $D'' \supset D$.

The existence of a family (X_D, P_μ) subject to conditions (1.7) and (1.11) is proved in [2].

It follows from (1.7)–(1.10) that

$$P_\mu\langle f, X_D\rangle = \langle K_D f, \mu\rangle. \tag{1.12}$$

1.3. Markov process (X_t, P_μ). Besides exit measures X_D we also consider random measures X_t which describe the mass distribution at a fixed time t. For every positive Borel function f,

$$P_\mu \exp\langle -f, X_t\rangle = \exp\langle -v_t, \mu\rangle, \tag{1.13}$$

where

$$v_t(x) + \Pi_x \int_0^t v_{t-s}(\xi_s)^\alpha\, ds = \Pi_x f(\xi_t) \tag{1.14}$$

with Random measures (X_t, P_μ) form a Markov process in the state space $\mathcal{M}$. This is a more traditional model of superdiffusion than the model described in Section 1.2.

[2]We denote by $\mathcal{M}(S)$ the space of all finite measures on a measurable space S.

1.4. Range and polar sets. Consider the class $\mathbb{C}$ of all closed random sets $C(\omega) \subset E \cup \partial E$ with the property that every exit measure X_D is concentrated, a.s.,[3] on C. There exists a minimal element of $\mathbb{C}$ and it is defined uniquely up to indistinguishability. We denote it by $\mathcal{R}$ and call it the *range of* X. A set $B \subset \partial E$ is called $\mathcal{R}$*-polar* if

$$P_x\{\mathcal{R} \cap B = \emptyset\} = 1 \qquad \text{for all } x \in D. \tag{1.15}$$

A Borel set B is $\mathcal{R}$-polar if and only if all compact subsets of B are $\mathcal{R}$-polar.

1.5. Principal results. Our objective is to study the class $\mathcal{U}$ of all positive solutions of the equation

$$Lu = u^\alpha \quad \text{in } E, \tag{1.16}$$

where E is a domain of class $C^{2,\lambda}$ in $\mathbb{R}^d$ and $1 < \alpha \leq 2$. We say that a solution u is *moderate* if it is dominated by an L-harmonic function. Every positive L-harmonic function has a unique representation

$$h(x) = \int_{\partial E} k(x,y)\nu(dy), \tag{1.17}$$

where k is the Poisson kernel for L and ν is a finite measure on ∂E.

The following result is proved in [7].

Theorem 1.1. *The formula*

$$u + \mathcal{E}(u) = h \tag{1.18}$$

establishes a 1-1 correspondence between moderate solutions u *and* L*-harmonic functions* h *such that the measure* ν *in (1.17) does not charge any* $\mathcal{R}$*-polar set. The function* h *is the minimal* L*-harmonic majorant of* u*, and* u *is the maximal solution dominated by* h*.*

We call ν the *trace of the moderate solution* u. In Section 3 we prove

Theorem 1.2. *If* u *is an arbitrary positive solution of (1.16) then, for every compact subset* B *of* ∂E*, there exists the maximal solution* u_B *dominated by* u *and equal to* 0 *on* $\partial E \setminus B$.[4] *There exist: (a) a maximal open subset* O *of* ∂E *such that* u_B *is moderate for every compact* $B \subset O$*; (b) a measure* ν *on* O *such that, for every compact* $B \subset O$*, the trace of* u_B *coincides with the restriction of* ν *to* B*.*

We call $\Gamma = \partial E \setminus O$ the *singular set of* u and we call the pair (Γ, ν) the *trace of* u.

We say that x is an *explosion point of a measure* ν if $\nu(U) = \infty$ for every neighborhood U of x.

Theorem 1.3. *A pair* (Γ, ν) *is the trace of a solution if and only if:*

(i) Γ *is compact;*

(ii) ν *is the Radon measure on* $O = \partial E \setminus \Gamma$*;*

(iii) ν *does not charge any* $\mathcal{R}$*-polar set;*

(iv) the conditions:

$$\begin{gathered}\Lambda \subset \Gamma \text{ is } \mathcal{R}\text{-polar and contains no explosion points of } \nu,\\ \Gamma \setminus \Lambda \text{ is closed}\end{gathered} \tag{1.19}$$

imply that $\Lambda = \emptyset$*.*

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[3] Writing a.s. means P_x-a.s. for all $x \in E$.

[4] Writing $u = f$ at $a \in \partial D$ means $u(x) \to f(a)$ as $x \to a, x \in D$. We write $u = f$ on $B \subset \partial D$ if $u = f$ at all $a \in B$.

Theorem 1.3 follows from 3.5.A and Theorems 5.3, 4.2 and 5.2.

A measure ν is called Σ-*finite* if there exist finite measures ν_n such that $\nu = \nu_1 + \cdots + \nu_n + \ldots$.

Theorem 1.4. *To every Σ-finite measure ν on ∂E which charges no $\mathcal{R}$-polar set there corresponds a continuous additive functional A^ν of the superdiffusion X. If (Γ, ν) satisfies conditions (i)–(iv) in Theorem 1.3, then*

$$u(x) = -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-A^\nu_\infty}\} \tag{1.20}$$

is the maximal element of $\mathcal{U}$ with the trace (Γ, ν).

Theorem 1.4 follows from Theorems 4.2, 4.3 and 6.1.

1.5. Bibliographical notes. The problem stated at the beginning of Section 1.4 has been studied recently by probabilistic and purely analytic methods. The work of analysts influenced the work of probabilists and vice versa.

It was shown in [2, Section 5.5] that an arbitrary positive solution of equation (1.16) has a representation of the form

$$u(x) = -\log P_x e^{-Z} \tag{1.21}$$

where Z is a functional of the superdiffusion X. Sufficient conditions on Z were established under which u given by (1.21) is a solution of (1.16). This way a 1-1 correspondence was established between all positive solutions of (1.16) and a certain class of functionals Z of X. [Analogous results in a parabolic setting are presented in more detail in [4, Section II.5].]

A boundary value problem

$$Lu = u^\alpha \quad \text{in } E, \qquad u = \nu \quad \text{on } \partial E \tag{1.22}$$

with a finite measure ν was discussed in [5, p.5]. In particular, it was conjectured that (1.22) has a solution if and only if ν does not charge any $\mathcal{R}$-polar set. The conjecture was proved in [7], where problem (1.22) was interpreted as the integral equation (1.18) with h given by (1.17).[5] The results of [7] (summarized in Theorem 1.1) describe all moderate solutions of (1.1).

It was proved in [6, Theorem 1.2a] that the class of $\mathcal{R}$-polar sets coincides with the class of null-sets of the Bessel capacity $\mathrm{Cap}^\partial_{2/\alpha,\alpha'}$. This implies that the condition

$$\alpha < \frac{d+1}{d-1} \tag{1.23}$$

holds if and only if the empty set is the only $\mathcal{R}$-polar set.

Condition (1.25) is satisfied if $\alpha = d = 2$. Le Gall [13] succeeded in describing all solutions of the equation $\Delta u = u^2$ in the unit disk in $\mathbb{R}^2$. He established a 1-1 correspondence between all solutions and all pairs (Γ, ν), where Γ is a closed subset of ∂E and ν is a Radon measure on $\partial E \setminus \Gamma$. The solution corresponding to (Γ, ν) is expressed in terms of the Brownian snake — a path-valued Markov process introduced in his earlier publications (this process is closely related to the super-Brownian motion). In [15], the results announced in [13] are proved in detail and are extended to a general smooth domain in $\mathbb{R}^2$. Recently Le Gall [16] extended some of the results of the present paper to a parabolic equation $\partial u/\partial t = \Delta u - u^2$ in a cylinder $\mathbb{R}_+ \times E$. He defined the trace of a positive solution u on $\{0\} \times E$ and

[5] In the particular case $\alpha = 2$, the conjecture was confirmed earlier by Le Gall [14].

proved propositions similar to our Theorems 1.3 and 1.4 [which he cited referring to our preprint].

A boundary value problem of type (1.22) was first considered by Gmira and Véron [12]. They investigated by purely analytic methods a class of functions ψ such that the problem

$$\Delta u = \psi(u) \quad \text{in } E, \qquad u = \nu \quad \text{on } \partial E \tag{1.24}$$

is solvable for every finite measure ν. This class contains $\psi(u) = u^\alpha$ with α subject to condition (1.23).

Marcus and Véron [17] investigated the equation $\Delta u = u^\alpha$, $\alpha > 1$, in the unit d-dimensional ball.[6] For every positive solution u they define the trace (Γ, ν) of u in terms of the boundary behavior of u.

Under condition (1.23), every pair (closed set Γ, Radon measure ν on $\partial E \backslash \Gamma$) is the trace of the uniquely defined positive solution u. (In the case $\alpha = d = 2$, this was proved earlier by Le Gall.) More results in the same direction are announced in [18]. In particular, the existence of a solution is stated, for $\alpha \geq \frac{d+1}{d-1}$, under conditions stronger than conditions (i)–(iv) in our Theorems 1.3 and 1.4. The authors have also investigated the parabolic case in a setting similar to those of [16].

Added in proof. The results announced in [17] and [18] are proved in two recent preprints [M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations*, I: *The subcritical case*; II: *The supercritical case*, Université François Rabelais, Tours]. In addition, they give necessary and sufficient conditions for a pair (Γ, ν) to be a trace in terms of a class of exceptional sets on the boundary and they extend the characterization of exceptional sets, given in [6] for $\alpha \leq 2$, to the case $\alpha > 2$. [Their method does not work for $\alpha \leq 2$.]

2. Equation $Lu = u^\alpha$. Operators V_D

2.1. Class $\mathcal{U}(D)$. We denote by $\mathcal{U}(D)$ the class of all positive functions u on D such that

$$Lu = u^\alpha \quad \text{in } D \tag{2.1}$$

and we put $\mathcal{U} = \mathcal{U}(E)$. We use the following facts:

2.1.A. (Comparison principle) If D is a bounded open set and if $u, v \geq 0$ satisfy the conditions

$$Lu - u^\alpha \geq Lv - v^\alpha \quad \text{in } D \tag{2.2}$$

and

$$\limsup_{x \to a, x \in D} [u(x) - v(x)] \leq 0 \qquad \text{for all } a \in \partial D, \tag{2.3}$$

then $u \leq v$ in D.

(See, e.g., Theorem 0.5 in [2].)

2.1.B. Suppose that $u_n \in \mathcal{U}(D)$ converge pointwise in D to u. Then $u \in \mathcal{U}(D)$. Let O be a relatively open subset of ∂D and let f be a continuous function on O. If all points of O are regular and if $u_n = f$ on O, then $u = f$ on O. (See Theorem 1.2 in [3].)

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[6] We emphasize that they cover the case $\alpha > 2$, which can not be investigated by using superdiffusions

2.1.C. Let X_D be the exit measure of an (L,α)-superdiffusion X from an open set D. Then, for every Borel function $f : \partial D \to [0,\infty]$,

$$u(x) = -\log P_x e^{-\langle f, X_D\rangle} \tag{2.4}$$

belongs to $\mathcal{U}(D)$, and $u = f$ at every regular point $a \in \partial D$ where f is continuous. If D is regular and if f is continuous on ∂D, then u is the minimal element of $\mathcal{U}(D)$ such that $u = f$ on ∂D. If, in addition, D is bounded and f is finite, then u is the unique element of $\mathcal{U}(D)$ such that $u = f$ on ∂D.

(See Theorems 1.2, 1.3 in [3] or Section II.3 in [4].)

2.1.D. (The mean value property) If $u \in \mathcal{U}(E)$ and if $\bar{D} \subset E$, then

$$u(x) = -\log P_x e^{-\langle u, X_D\rangle} \quad \text{in } D. \tag{2.5}$$

(See Lemma 1.1 in [3] or Theorem 2.3 in [6].)

2.1.E. Suppose that $D_1 \supset D_2$ and $\Gamma \cap D_1 = \emptyset$. Then $X_{D_1}(\Gamma) \geq X_{D_2}(\Gamma)$ P_μ-a.s. for every μ.

(See Lemma 3.1 in [8].)

2.2. Operators V_D. With every open $D \subset E$ we associate an operator which acts on positive Borel functions by the formula

$$V_D(f)(x) = -\log P_x e^{-\langle f_E, X_D\rangle}, \tag{2.6}$$

where $f_E = 1_E f$. If $x \notin D$, then $P_x\{X_D = \delta_x\} = 1$ and therefore $V_D(f) = f$ on $E \setminus D$. For every positive Borel f, $V_D(f) \in \mathcal{U}(D)$. [This follows, for instance, from Theorems 2.1-2.2 in [6].] Formula (1.8) implies that

$$V_D(f) \leq K_D f. \tag{2.7}$$

The operators V_D have also the following properties:

2.2.A. $V_D(f_1) \leq V_D(f_2)$ for $f_1 \leq f_2$.

2.2.B. If $D_1 \subset D_2$, then $V_{D_1}V_{D_2} = V_{D_2} = V_{D_2}V_{D_1}$.

Proof. Put $V_i = V_{D_i}$ and $v_i = V_i(f)$. By the Markov property (1.11),

$$v_2(x) = -\log P_x P_{X_{D_1}} e^{-\langle f, X_{D_2}\rangle} = -\log P_x e^{-\langle v_2, X_{D_1}\rangle} = V_1(v_2)(x).$$

The equality $V_2(v_1) = v_2$ holds because $v_1 = f$ on $E \setminus D_1$, and $V_2(g)$ depends only on the values of g on $E \setminus D_2$. □

2.2.C. For every f_1, f_2,

$$V_D(f_1 + f_2) \leq V_D(f_1) + V_D(f_2).$$

Proof. Let $u_i = V_D(f_i), i = 1, 2$, and $u = V_D(f_1 + f_2)$. By (1.8),

$$u_i + \mathcal{E}_D(u_i) = K_D f_i, \qquad u + \mathcal{E}(u) = K_D f_1 + K_D f_2.$$

We have $(u_1 + u_2)^\alpha = u_1^\alpha + u_2^\alpha + \varphi$, where $\varphi \geq 0$. Note that $u_1 + u_2 + \mathcal{E}(u_1 + u_2) = u_1 + u_2 + \mathcal{E}(u_1) + \mathcal{E}(u_2) + G_D\varphi = K_D(f_1 + f_2) + G_D\varphi = u + \mathcal{E}(u) + G_D\varphi$. By Theorem 2.1 in [8], this implies $u_1 + u_2 \geq u$. □

2.2.D. For every D_1, D_2,

$$V_{D_1 \cap D_2}(f) \leq V_{D_1}(f) + V_{D_2}(f) \quad \text{in } D_1 \cap D_2.$$

Proof. Put $D = D_1 \cap D_2$. Note that $\partial D = B_1 \cup B_2$, where $B_1 \subset \partial D_1$, $B_2 \subset \partial D_2$ and $B_1 \cap B_2 = \emptyset$. Put $f_i = 1_{B_i} f$. We have $f = f_1 + f_2$ on ∂D, and therefore $V_D(f) = V_D(f_1 + f_2)$ on D. By 2.2.C,

$$V_D(f) \le V_D(f_1) + V_D(f_2) \quad \text{on } D. \tag{2.8}$$

By 2.1.E, $X_D(\Gamma) \le X_{D_1}(\Gamma)$ for all $\Gamma \subset B_1$. Hence

$$\langle f_1, X_D \rangle = \int_{B_1} f dX_D \le \int_{B_1} f dX_{D_1} \le \langle f, X_{D_1} \rangle,$$

which implies $V_D(f_1) \le V_{D_1}(f)$. Analogously, $V_D(f_2) \le V_{D_2}(f)$, and (2.8) implies 2.2.D. □

2.3. Action of V_D on $\mathcal{U}$.

Theorem 2.1. *Suppose that D is a regular open subset of E and $u \in \mathcal{U}$. Then*

$$V_D(u) \le u. \tag{2.9}$$

If $D_1 \subset D_2$, then

$$V_{D_2}(u) \le V_{D_1}(u). \tag{2.10}$$

Proof. Let $v = V_D(u)$. Note that $\partial D = A \cup B$, where $A = \partial D \cap \partial E$ and $B = \partial D \cap E$. Denote by A_0 the set of all points $a \in A$ at a positive distance from B. By 2.1.C, $v = u$ on B and $v = 0$ on A_0. Consider a sequence of continuous functions f_n on ∂D such that $f_n \uparrow u$ on B and $f_n = 0$ on A. Let $v_n(x) = -\log P_x e^{-\langle f_n, X_D \rangle}$. By 2.1.C, $v_n \in \mathcal{U}(D)$ and $v_n = f_n$ on ∂D. Therefore

$$\limsup_{x \to a, x \in D} [v_n(x) - u(x)] \le 0$$

for all $a \in \partial D$. By 2.1.A, $v_n \le u$, and (2.9) holds because $v_n \uparrow v$.

If $D_1 \subset D_2$, then, by 2.2.B, (2.9) and 2.2.A, $V_{D_2}(u) = V_{D_1} V_{D_2}(u) \le V_{D_1}(u)$. □

3. Operators Q_B

3.1. Construction. For every closed set $B \subset \partial E$ and for every $\varepsilon > 0$ we put

$$D(B, \varepsilon) = \{x \in E : d(x, B) > \varepsilon\}. \tag{3.1}$$

The open sets (3.1) are regular. If $u \in \mathcal{U}$, then by Theorem 2.1, $V_{D(B,\varepsilon)}(u)$ is monotone increasing in ε and therefore there exists a limit

$$Q_B(u) = \lim_{\varepsilon \to 0} V_{D(B,\varepsilon)}(u). \tag{3.2}$$

It follows from 2.1.B that $Q_B(u) \in \mathcal{U}$. The operators Q_B have the following properties:

3.1.A. $Q_B(u_1) \le Q_B(u_2)$ for $u_1 \le u_2$.

3.1.B. $Q_B(u) \le u$.

3.1.C. For every B_1, B_2, $\quad Q_{B_1 \cup B_2}(u) \le Q_{B_1}(u) + Q_{B_2}(u)$.

3.1.D. $Q_{\partial E}(u) = u$.

3.1.E. If $u \le u_1 + u_2$, then $Q_B(u) \le Q_B(u_1) + Q_B(u_2)$.

Property 3.1.A is an implication of 2.2.A; 3.1.B follows from (2.9); 3.1.C is an implication of 2.2.D and the relation $D(B_1 \cup B_2, \varepsilon) = D(B_1, \varepsilon) \cap D(B_2, \varepsilon)$. 3.1.D holds because, by 2.1.D, $V_{D(\partial E,\varepsilon)}(u) = u$ for all ε. Finally, 3.1.E follows from 2.2.C and 2.2.A.

3.2. Extremal characterization.

Theorem 3.1. *For every $u \in \mathcal{U}$, the function $u_B = Q_B(u)$ is the maximal element of $\mathcal{U}$ subject to the conditions*

$$u_B \le u; \quad u_B = 0 \quad \text{on } \partial E \setminus B. \tag{3.3}$$

Proof. Put $D^\varepsilon = D(B,\varepsilon), B^\varepsilon = \partial D_\varepsilon \cap E$ and denote by A^ε the set of all $x \in \partial E$ at a distance $> \varepsilon$ from B. By 2.1.C, the function $v^\varepsilon = V_{D^\varepsilon}(u)$ belongs to $\mathcal{U}(D^\varepsilon)$ and satisfies the conditions $v^\varepsilon = 0$ on A^ε, $v^\varepsilon = u$ on B^ε. It follows from 3.1.B, (3.2) and 2.1.B that u_B belongs to $\mathcal{U}$ and satisfies (3.3).

Suppose (3.3) holds for $\tilde{u}$. Then $\tilde{u} \le u = v^\varepsilon$ on B^ε and $\tilde{u} = 0 \le v^\varepsilon$ on $\partial D^\varepsilon \setminus B^\varepsilon \subset \partial E \setminus B$. By 2.1.A, $\tilde{u} \le v^\varepsilon$ in D^ε, and therefore $\tilde{u} \le u_B$. □

Theorem 3.1 implies:

3.2.A. If $B_1 \supset B_2$, then $Q_{B_1}Q_{B_2} = Q_{B_2} = Q_{B_2}Q_{B_1}$ and $Q_{B_1}(u) \ge Q_{B_2}(u)$.

3.2.B. If B and B_0 are disjoint compact subsets of ∂E, then $Q_B Q_{B_0} = 0$.

Indeed, $Q_B[Q_{B_0}(u)] = 0$ on $\partial E \setminus B$ and $Q_B[Q_{B_0}(u)] \le Q_{B_0}(u) = 0$ on $\partial E \setminus B_0$.

3.3. Probabilistic representation. We say that a sequence of open sets D_n is a standard sequence approximating E if their closures $\bar{D}_n$ are compact, $\bar{D}_n \subset D_{n+1}$ and $D_n \uparrow E$. It is proved in Section II.5 of [4] that, for every $u \in \mathcal{U}$, there exists a function Z such that

$$Z = \lim \langle u, X_{D_n} \rangle \quad P_\mu\text{-a.s.} \tag{3.4}$$

for every standard sequence D_n and every $\mu \in \mathcal{M}(E)$. Since $V_{D_n}(u) = u$ in D_n, the dominated convergence theorem implies that

$$u(x) = -\log P_x e^{-Z}. \tag{3.5}$$

By applying this result to $u_B = Q_B(u)$, we get a probabilistic representation of the operators Q_B:

$$Q_B(u)(x) = -\log P_x e^{-Z_B}, \tag{3.6}$$

where

$$Z_B = \lim \langle u_B, X_{D_n} \rangle \quad P_\mu\text{-a.s.} \tag{3.7}$$

It follows from 3.2.A and 3.1.C that

3.3.A. If $B_1 \supset B_2$, then $Z_{B_1} \ge Z_{B_2}$ a.s.

3.3.B. For every B_1, B_2, $Z_{B_1 \cup B_2} \le Z_{B_1} + Z_{B_2}$ a.s.

We need another representation of Z_B:

3.3.C.[7] For every sequence $\varepsilon_n \downarrow 0$ and every $\mu \in \mathcal{M}(E)$,

$$Z_B = \lim \langle u, X_{D(B,\varepsilon_n)} \rangle \quad P_\mu\text{-a.s.} \tag{3.8}$$

Proof. 1°. Put $\tilde{D}_n = D(B,\varepsilon_n), Y_n = e^{-\langle u, X_{\tilde{D}_n} \rangle}$. By (1.11), (1.7), (2.6) and (2.9),

$$P_\mu\{Y_{n+1} | \mathcal{F}_{\subset \tilde{D}_n}\} = P_{X_{\tilde{D}_n}} Y_{n+1} = e^{-\langle V_{\tilde{D}_{n+1}}(u), X_{\tilde{D}_n} \rangle} \ge Y_n \quad P_\mu\text{-a.s.}$$

Hence $(Y_n, \mathcal{F}_{\subset \tilde{D}_n}, P_\mu)$ is a bounded submartingale which implies the existence, a.s., of the limit

$$\tilde{Z}_B = \lim \langle u, X_{\tilde{D}_n} \rangle. \tag{3.9}$$

[7] By $\langle u, X_D \rangle$ we mean the integral of u over $D^c \cap E$. In other words, we set $u = 0$ on ∂E.

By (3.2) and the dominated convergence theorem,

$$P_\mu e^{-\tilde{Z}_B} = e^{-\langle u_B,\mu\rangle}. \tag{3.10}$$

2°. By 1° applied to u_B, we get the existence of the limit

$$\hat{Z}_B = \lim \langle u_B, X_{\tilde{D}_n}\rangle \tag{3.11}$$

and the equation

$$P_\mu e^{-\hat{Z}_B} = e^{-\langle Q_B(u_B),\mu\rangle}. \tag{3.12}$$

By 3.2.A, $Q_B(u_B) = u_B$, and (3.12) and (3.10) yield

$$P_\mu e^{-\hat{Z}_B} = P_\mu e^{\tilde{Z}_B}. \tag{3.13}$$

By 3.1.B, $\hat{Z}_B \le \tilde{Z}_B$, and (3.13), (3.11) imply

$$\tilde{Z}_B = \hat{Z}_B = \lim\langle u_B, X_{\tilde{D}_n}\rangle \quad \text{a.s.} \tag{3.14}$$

3°. Fix $\varepsilon > 0$. By (3.3), $u_B = 0$ on $\partial E \setminus B$. Therefore there exists $\delta_n > 0$ such that $u_B(x) \le \varepsilon$ if $x \in \tilde{D}_n$ and $d(x, \partial E) \le \delta_n$. Formula (3.7) holds for a standard sequence $D_n = \{x : d(x, B) > \varepsilon_n, d(x, \partial E) > \delta_n\}$ approximating E. We have $D_n = D^*_n \cap \tilde{D}_n$, where $D^*_n = \{x : d(x, \partial E) > \delta_n\}$, and, by 2.1.E,

$$\langle u_B, X_{D_n}\rangle \le \langle u_B, X_{\tilde{D}_n}\rangle + \langle \varepsilon, X_{D^*_n}\rangle. \tag{3.15}$$

It follows from (3.15), (3.7) and (3.14) that

$$Z_B \le \tilde{Z}_B + \varepsilon \lim\langle 1, X_{D^*_n}\rangle. \tag{3.16}$$

By [7, Lemma 1.2], for every L-harmonic function h and for every standard sequence D_n, there exists, P_μ-a.s., a finite limit $\lim\langle h, X_{D_n}\rangle$, and the limit does not depend, P_μ-a.s., on D_n. This is applicable to $h = 1$ and $D_n = D^*_n$; and, by letting $\varepsilon \to 0$ in (3.16), we get $Z_B \le \tilde{Z}_B$. Equations (3.6) and (3.10) imply $Z_B = \tilde{Z}_B$. □

3.4. Trace of a moderate solution. It follows from Theorems 1.3, 2.1 and 3.1 in [7] and Theorem 1.2a in [6] that:

3.4.A. If u is moderate, then

$$h = u + \mathcal{E}(u) \tag{3.17}$$

is the minimal L-harmonic majorant of u, and u is the maximal solution dominated by h.

3.4.B. Let h be an L-harmonic function with trace ν. Equation (3.17) has a solution u if and only if ν does not charge $\mathcal{R}$-polar sets.

3.4.C. If (3.17) holds for an L-harmonic h, then $u \in \mathcal{U}$.

The *trace of a moderate solution* u is defined as the trace of the L-harmonic function $h = u + \mathcal{E}(u)$. By 3.4.B, a finite measure ν is a trace if and only if $\nu(B) = 0$ for all $\mathcal{R}$-polar sets B. It follows from 3.4.A that the inequality $u_1 \le u_2$ between two moderate solutions is equivalent to the inequality $h_1 \le h_2$ between their minimal L-harmonic majorants, which is equivalent to the inequality $\nu_1 \le \nu_2$ between the traces.

We use the following facts. Let k be the Poisson kernel in E. If h is a positive L-harmonic function with trace ν, then there exists a measure Π^h_x on the space of paths killed at the first exit from E such that, for every $B \subset \partial E$,

$$\Pi^h_x\{\xi_\zeta \in B\} = \int_B k(x,y)\nu(dy), \tag{3.18}$$

and, for every stopping time τ,

$$\Pi_x^h\{\tau < \zeta\} = \Pi_x h(\xi_\tau). \tag{3.19}$$

[The probability measure $\frac{1}{h(x)}\Pi_x^h$ is called the h-transform of Π_x.] Moreover,

$$\Pi_x^h = \int \Pi_x^y \nu(dy), \tag{3.20}$$

where Π_x^y is the measure corresponding to the L-harmonic function $k(\cdot, y)$.

Theorem 3.2. *If u is a moderate solution, then, for every B, $u_B = Q_B(u)$ is also moderate and the trace of u_B is the restriction of the trace of u to B.*

Proof. The first part follows from 3.1.B.

Let ν be the trace of u. Consider the minimal L-harmonic majorant

$$h(x) = \int_{\partial E} k(x,y)\nu(dy)$$

of u and the minimal L-harmonic majorant h_B of u_B. Put

$$\hat{h}_B(x) = \int_B k(x,y)\nu(dy).$$

By (3.18) and (3.19),

$$\hat{h}_B(x) = \Pi_x^h\{\xi_\zeta \in B\} = \lim_{\varepsilon\to 0}\Pi_x^h\{\tau_\varepsilon < \zeta\} = \lim_{\varepsilon\to 0}\Pi_x h(\xi_{\tau_\varepsilon}), \tag{3.21}$$

where τ_ε is the first exit time from $D(B,\varepsilon)$. By (2.7),

$$V_{D(B,\varepsilon)}(u)(x) \le \Pi_x u(\xi_{\tau_\varepsilon}) \le \Pi_x h(\xi_{\tau_\varepsilon}). \tag{3.22}$$

By (3.2), (3.21) and (3.22), $u_B \le \hat{h}_B$, which implies

$$h_B \le \hat{h}_B. \tag{3.23}$$

By 3.4.B, there exists $\hat{u}_B$ such that $\hat{u}_B + \mathcal{E}(\hat{u}_B) = \hat{h}_B$. Note that $\hat{u}_B \le u$ and $\hat{u}_B \le \hat{h}_B = 0$ on $\partial E \setminus B$. By Theorem 3.1, $u_B \ge \hat{u}_B$, and therefore $h_B \ge \hat{h}_B$. By (3.23), $h_B = \hat{h}_B$. □

3.5. General definition of trace. Fix a solution u. We say that a compact set $B \subset \partial E$ is *moderate for u* if the solution $u_B = Q_B(u)$ is moderate. Let ν_B stand for the trace of u_B. By 3.1.C, the union of two moderate sets is moderate. Suppose that B is moderate and let $\tilde{B} \subset B$. By 3.2.A, $u_{\tilde{B}} \le u_B$ and therefore $\tilde{B}$ is moderate. It follows from Theorem 3.2 and 3.2.A that $\nu_{\tilde{B}}$ is the restriction of ν_B to $\tilde{B}$.

A relatively open subset A of ∂E is called moderate if all compact subsets of A are moderate. The union O of all moderate open sets is moderate. Clearly, there exists a unique measure ν on O such that its restriction to an arbitrary compact subset B coincides with ν_B. This measure is a Radon measure on O (that is, it is finite on all compact subsets). We call the closed set $\Gamma = \partial E \setminus O$ the *singular set of the solution u* and we call the pair (Γ, ν) the *trace of u on ∂E*. A solution u is moderate if and only if the singular set is empty (in this case $\nu(\partial E) < \infty$).

If a compact set B is $\mathcal{R}$-polar, then $Q_B(u) = 0$. This follows from the relation

$$\{\mathcal{R} \cap B = \emptyset\} = \bigcup_\varepsilon \{X_{D(B,\varepsilon)} = 0\}$$

(see, e.g., the proof of Theorem 4.2 in [4]). Hence $\mathcal{R}$-polar sets are moderate for all u. We conclude from 3.4.A,B,C that:

3.5.A. If (Γ, ν) is the trace of a solution u, then ν does not charge $\mathcal{R}$-polar sets.

4. Solutions determined by continuous linear additive functionals

4.1. Continuous linear additive functionals. Let X be a superdiffusion in E and let $\mathcal{M}^*$ be a set of finite measures on E which contains all Dirac measures $\delta_x, x \in E$. We assume that $\mathcal{M}^*$ contains, with every μ, all measures $\tilde{\mu} \le \mu$ and that $P_\mu\{X_D \in \mathcal{M}^*\} = 1$ for all D and all $\mu \in \mathcal{M}^*$. Denote by $\mathcal{F}$ the σ-algebra generated by $X_s, s < \infty$, and by $\mathcal{F}_t$ the σ-algebra generated by $X_s, s \le t$. A function $A_t(\omega)$ from $[0,\infty] \times \Omega$ to $[0,\infty)$ is called a *finite continuous additive functional of X with determining set $\mathcal{M}^*$* if:

4.1.A. $A_0 = 0$.

4.1.B. For every t and every μ, A_t is measurable with respect to the P_μ-completion of $\mathcal{F}$; it is also measurable with respect to P_μ-completion of $\mathcal{F}_t$ if $\mu \in \mathcal{M}^*$.

4.1.C. $A_{s+t} = A_s + \theta_s A_t$ P_μ-a.s. for all $\mu \in \mathcal{M}^*$ and all pairs s, t. (Here θ_s are the shift operators for X.)

4.1.D. A_t is continuous in t for P_μ-almost all ω for every $\mu \in \mathcal{M}^*$.

A *continuous additive functional with determining set* $\mathcal{M}^*$ is a function $A_t(\omega)$ which can be represented as the sum of a countable family of finite continuous additive functionals with the same determining set. [If $\sigma = \sup\{t : A_t < \infty\}$, then A_t is continuous on $[0,\sigma)$ and $A_t = \infty$ for $t > \sigma$.] To every functional A_t there corresponds a diffuse measure[8] $a(dt)$ on $(0,\infty]$ such that $a(0,t] = A_t$ for all t. This measure is finite if and only the functional A is finite.

If A^n are continuous additive functionals with determining sets $\mathcal{M}^*_n$ and if $\mathcal{M}^* \subset \mathcal{M}^*_n$ for all n, then $A_t = A^1_t + \dots + A^n_t + \dots$ is a continuous additive functional with determining set $\mathcal{M}^*$.

Let A and $\tilde{A}$ be continuous additive functionals with determining sets $\mathcal{M}^*$ and $\tilde{\mathcal{M}}^*$. We say that A and $\tilde{A}$ are *equivalent*, and write $A \sim \tilde{A}$, if $P_\mu\{A_t = \tilde{A}_t\} = 1$ for all t and all $\mu \in \mathcal{M}^* \cap \tilde{\mathcal{M}}^*$. We say that A and $\tilde{A}$ are *indistinguishable* if, in addition, $\mathcal{M}^* = \tilde{\mathcal{M}}^*$.

We call

$$h(x) = P_x A_\infty \tag{4.1}$$

the *potential* and

$$u(x) = -\log P_x e^{-A_\infty} \tag{4.2}$$

the *log-potential* of A. We say that a continuous additive functional A is *linear* if

$$P_\mu A_\infty = \langle h, \mu \rangle \tag{4.3}$$

and

$$P_\mu e^{-A_\infty} = e^{-\langle u, \mu \rangle} \tag{4.4}$$

for all μ in the determining set $\mathcal{M}^*$.

[8] A measure a is called diffuse if it does not charge single points.

Suppose that the potential h of a continuous linear additive (CLA) functional A is an L-harmonic function h. It follows from Theorem 1.2 in [9] that the log-potential u of A satisfies the equation

$$u + \mathcal{E}(u) = h \quad \text{in } E. \tag{4.5}$$

Moreover, by Theorem 2.1 in [8], u is the unique solution of (4.5). By 3.4.C, u satisfies

$$Lu = u^\alpha \quad \text{in } E. \tag{4.6}$$

Clearly, u is a moderate solution of (4.6).

4.2. Classes H^* and H^{**}. Denote by H^* the class of all positive L-harmonic functions h such that equation (4.5) has a positive solution u. Put $h \in H^{**}$ if $h \in H^*$ and if $\mathcal{E}(h)(x) < \infty$ for some x (which implies that $\mathcal{E}(h)(x) < \infty$ for all x). It follows from [7, Theorem 3.1 and proof of Theorem 2.2] (cf. [9, 4.1.B]) that every $h \in H^*$ can be represented in the form

$$h = h_1 + \dots + h_n + \dots \quad \text{with } h_n \in H^{**}. \tag{4.7}$$

Theorems 1.7, 1.1, 1.3 in [10] imply

Theorem 4.1. *If $h \in H^{**}$, then there exists a unique (up to indistinguishability) continuous linear additive functional A of X with potential h and determining set $\{\mu : \langle h + \mathcal{E}(h), \mu\rangle < \infty\}$. The log-potential u of A is determined uniquely by (4.5).*

Remark. If A^n are CLA functionals corresponding to $h_n \in H^{**}$ by Theorem 4.1 and if $h = h_1 + \dots + h_n + \dots \in H^{**}$, then $A^1 + \dots + A^n + \dots$ is equivalent to the CLA functional corresponding to h.

4.3. Continuous linear additive functional with spectral measure ν. Denote by $\mathcal{N}^*$ the set of all measures ν on ∂E such that $K\nu \in H^*$. Analogously, $\nu \in \mathcal{N}^{**}$ if $K\nu \in H^{**}$. Put $\nu \in \mathcal{N}$ if there exist $\nu_n \in \mathcal{N}^{**}$ such that

$$\nu = \nu_1 + \dots + \nu_n + \dots . \tag{4.8}$$

Note that $\mathcal{N} \supset \mathcal{N}^*$.

Remark 1. Clearly, every $\nu \in \mathcal{N}$ is Σ-finite.

Remark 2. If $\nu^n \in \mathcal{N}^{**}$ and $\nu^n \uparrow \nu \in \mathcal{N}^{**}$, then, by the Remark to Theorem 4.1, $\lim A^{\nu_n} \sim A^\nu$.

Theorem 4.2. *It is possible to define for every $\nu \in \mathcal{N}$ a continuous linear additive functional A^ν of X with determining set $\mathcal{M}^*(\nu)$ in such a way that:*

4.3.A. If $\nu = \nu_1 + \dots + \nu_n + \dots$, then A^ν is equivalent to $A^{\nu_1} + \dots + A^{\nu_n} + \dots$.

4.3.B. If $\nu \in \mathcal{N}^{**}$ and if $h = K\nu$, then A^ν is equivalent to the CLA functional corresponding to $h = K\nu$ by Theorem 4.1.

4.3.C. For every $\nu \in \mathcal{N}$, the log-potential u of A^ν satisfies the equation (4.6).

Proof. 1°. We use Theorem 4.1 to define A^ν for $\nu \in \mathcal{N}^{**}$, and we define A^ν as the sum of A^{ν_n} if ν is defined by (4.8). To justify this definition, we prove that, if $\nu = \sum \nu_i = \sum \tilde{\nu}_j$, then $\sum A^{\nu_i} \sim \sum A^{\tilde{\nu}_j}$. Indeed, since the measure ν is Σ-finite, there exists a finite measure μ such that $\nu(dx) = \rho(x)\mu(dx)$ for a positive function ρ. Put

$$\nu^m = \nu_1 + \dots + \nu_m, \qquad \tilde{\nu}^n = \tilde{\nu}_1 + \dots + \tilde{\nu}_n.$$

By the Radon-Nikodým theorem, $\nu^m(dx) = \rho^m(x)\mu(dx)$, $\tilde{\nu}^n(dx) = \tilde{\rho}^n(x)\mu(dx)$ with $\lim \rho^m = \lim \tilde{\rho}^n = \rho$ μ-a.e. The measures $\nu^{mn}(dx) = [\rho^m(x) \wedge \tilde{\rho}^n(x)]\mu(dx)$ belong to $\mathcal{N}^{**}$. The corresponding functionals $A^{mn} = A^{\nu^{mn}}$ are monotone increasing in m and n, and therefore

$$\lim_m \lim_n A^{mn} = \lim_n \lim_m A^{mn} = \sup_{m,n} A^{mn}. \tag{4.9}$$

Since $\nu^{mn} \uparrow \nu^m$ as $n \to \infty$, we have $\lim_n A^{mn} \sim A^{\nu^m}$ by Remark 2. Clearly, $A^{\nu^m} \sim A^{\nu_1} + \cdots + A^{\nu_m}$, and therefore the left side in (4.9) is equal to $\sum A^{\nu_i}$. Analogously, the right side is equal to $\sum \tilde{A}^{\nu_j}$.

2°. Properties 4.3.A and 4.3.B hold for $\nu \in \mathcal{N}^{**}$ by Theorem 4.1, and they can be obtained for $\nu \in \mathcal{N}$ by a passage to the limit.

3°. If $\nu_k \in \mathcal{N}^{**}$, then $\nu^n = \nu_1 + \cdots + \nu_n \in \mathcal{N}^{**}$ for all n, and the log-potential u_n of A^ν satisfies (4.4) by Theorem 4.1 and (4.6) by 3.4.C. The function $u = \lim u_n$ satisfies (4.6) by 2.1.B. We get (4.4) by a passage to the limit. □

We call A^ν the *CLA functional with spectral measure* ν.

Theorem 4.3. *A Σ-finite measure ν belongs to $\mathcal{N}$ if and only if it charges no $\mathcal{R}$-polar sets.*

Proof. For a finite measure ν, this follows from 3.4.B. The extension to Σ-finite ν is obvious. □

Theorem 4.4. *Let $\nu \in \mathcal{N}$. The following three conditions are equivalent:*
(i) ν is finite;
(ii) the potential h of A^ν is L-harmonic;
(iii) the log-potential u of A^ν is a moderate solution of (4.6).

Proof. The equivalence of (i) and (ii) follows from the relation

$$h(x) = P_x A^\nu_\infty = \int k(x,y)\nu(dy),$$

which follows from 4.3.B if $\nu \in \mathcal{N}^{**}$ and which can be obtained by a monotone passage to the limit for an arbitrary $\nu \in \mathcal{N}$.

Suppose that $\nu_n \in \mathcal{N}^{**}$ and $\nu_n \uparrow \nu$. The potential h_n and the log-potential u_n of A^{ν_n} are related by equation $u_n + \mathcal{E}(u_n) = h_n$. Clearly, $h_n \uparrow h$, $u_n \uparrow u$, and therefore $u + \mathcal{E}(u) = h$. If h is L-harmonic, then u is moderate. On the other hand, if u is dominated by an L-harmonic $\tilde{h}$, then $u_n \le \tilde{h}$, which implies $K\nu_n \le \tilde{h}$. We conclude that $K\nu \le \tilde{h}$, and therefore ν is finite. □

4.4. Log-potentials of functionals A^ν.

4.4.A. The log-potential u of A^ν vanishes on ∂E outside the support of ν.

Proof. Let $\nu(U) = 0$ for an open subset U of ∂E. If ν_n are defined by (4.8), then $\nu_n(U) = 0$. We have $u_n + \mathcal{E}(u_n) = h_n$ where h_n is the potential and u_n is the log-potential of A^{ν_n}. By (1.5), $h_n = 0$ on U. Hence $u_n = 0$ on U and $u = \lim u_n = 0$ on U by 2.1.B. □

4.4.B. If u_1, u_2, u are the log-potentials of functionals $A^{\nu_1}, A^{\nu_2}, A^{\nu_1+\nu_2}$, then $u \le u_1 + u_2$.

Proof. If $\nu_1, \nu_2 \in \mathcal{N}^{**}$, then $u_i + \mathcal{E}(u_i) = K\nu_i < \infty$ and $u + \mathcal{E}(u) = K(\nu_1 + \nu_2)$. We get $u \le u_1 + u_2$ by the same computation as in proof of 2.2.C. The general case is covered by a monotone passage to the limit. □

4.4.C. Suppose that $\nu \in \mathcal{N}$ is finite and is concentrated on a closed set $B \subset \partial E$. Let h be the potential and u be the log-potential of A^ν. If $\varepsilon_n \downarrow 0$, then

$$A^\nu_\infty = \lim \langle u, X_{D(B,\varepsilon_n)} \rangle \quad P_\mu\text{-a.s.} \tag{4.10}$$

for all $\mu \in \mathcal{M}^*(\nu)$.

Proof. By Theorem 4.4, u is a moderate solution with trace ν. It follows from Theorem 3.2 that $u_B = Q_B(u)$ has the same trace as u, and therefore $u_B = u$. By 3.3.C,

$$Z = \lim \langle u, X_{D_n} \rangle = \lim \langle u, X_{D(B,\varepsilon_n)} \rangle \quad \text{a.s.} \tag{4.11}$$

for an arbitrary standard sequence D_n. By (4.5), $u \le h$, and, by Theorem 5.1 in [9],

$$A^\nu_\infty = \lim \langle h, X_{D_n} \rangle \ge Z \quad P_\mu\text{-a.s.} \tag{4.12}$$

if $\mu \in \mathcal{M}^*(\nu)$. By (3.5), and (4.2),

$$u(x) = -\log P_x e^{-Z} = -\log P_x e^{-A^\nu_\infty}. \tag{4.13}$$

Formula (4.10) follows from (4.11), (4.12) and (4.13). □

4.4.D. Suppose that $\nu \in \mathcal{N}$ vanishes on a closed set B and is finite on a neighborhood U of B. If u is the log-potential of A^ν, then $Q_B(u) = 0$.

Proof. Denote by ν_1, ν_2 the restrictions of ν to U and U^c, respectively. Let u_1 and u_2 be the log-potentials of A^{ν_1} and A^{ν_2}. By 4.4.B, $u \le u_1 + u_2$ and, by 3.1.E, $Q_B(u) \le Q_B(u_1) + Q_B(u_2)$.

By Theorem 4.4, u_1 is a moderate solution. Its trace ν_1 vanishes on B, and $Q_B(u_1) = 0$ by Theorem 3.2.

By 4.4.A, $u_2 = 0$ on U, and we conclude from Theorem 3.1 that $Q_B(u_2) = 0$. □

5. (Γ, ν)-SOLUTIONS

5.1. Let O be an open set on the boundary ∂E and let ν be a measure on ∂E which charges no $\mathcal{R}$-polar set. We put $\nu \in \mathcal{N}(O)$ if ν is concentrated on O and if its restriction to O (which we denote again by ν) is a Radon measure on O. All measures $\nu \in \mathcal{N}(O)$ are Σ-finite and, by Theorem 4.3, $\mathcal{N}(O) \subset \mathcal{N}$. Let Γ be a closed subset of ∂E. By Theorem 2.1 in [3],

$$w(x) = -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset\} \tag{5.1}$$

is the maximal element of $\mathcal{U}$ such that

$$w = 0 \quad \text{on } O = \partial E \setminus \Gamma. \tag{5.2}$$

For every $\nu \in \mathcal{N}(O)$,

$$u(x) = -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-A^\nu_\infty}\} \tag{5.3}$$

is also an element of $\mathcal{U}$. [This is proved in Section 4 of [11] by using the probabilistic description of $\mathcal{U}$ given in Section 0.8 of [5].] We call u the (Γ, ν)-*solution.* Our objective is to evaluate the trace of such solution. To simplify notation, we drop the superscript ∞ in (5.3) and similar formulae. It follows from (5.3) and Theorem II.4.3 in [4] that

$$P_\mu\{\mathcal{R}\cap\Gamma=\emptyset, e^{-A^\nu}\}=e^{-\langle u,\mu\rangle} \tag{5.4}$$

if the support of the measure μ is disjoint from Γ.

5.2.

Theorem 5.1. *Let u be the (Γ,ν)-solution and let ν' be the restriction of ν to a closed subset B of O. Then $Q_B(u)$ is equal to the log-potential of $A'=A^{\nu'}$, and*

$$Z_B=A'_\infty \quad P_x\text{-a.s.} \tag{5.5}$$

for all x. (Here Z_B is given by (3.7) or (3.8).)

Proof. Let $A''=A^{\nu''}$, where ν'' is the restriction of ν to $B''=O\setminus B$. Denote by u',u'' the log-potentials of A',A'', and put $D_\varepsilon=D(B,\varepsilon)$. By (2.6),

$$u_\varepsilon(x)=V_{D_\varepsilon}(u)(x)=-\log P_x e^{-\langle u,X^\varepsilon\rangle},$$

where X^ε is the restriction to E of the exit measure from D_ε. By (5.4), this implies

$$u_\varepsilon(x)=-\log P_xP_{X^\varepsilon}\{\mathcal{R}\cap\Gamma=\emptyset,e^{-A^\nu}\}.$$

By 4.4.C, A' is measurable with respect to the P_x-completion of $\mathcal{F}_{\supset D_\varepsilon}$; and, by (1.11), $P_xe^{-A'}=P_xP_{X^\varepsilon}e^{-A'}$. By (2.6) and (5.4),

$$e^{-V_{D^\varepsilon}(u'')(x)}=P_xe^{-\langle u'',X^\varepsilon\rangle}=P_xP_{X^\varepsilon}e^{-A''}.$$

By (5.4),

$$P_{X^\varepsilon}\{\mathcal{R}\cap\Gamma\neq\emptyset\}=1-e^{-\langle w,X^\varepsilon\rangle}\le\langle w,X^\varepsilon\rangle,$$

where w is given by (5.1). Hence,

$$\begin{aligned}|e^{-u'(x)}-e^{-u_\varepsilon(x)}|&\le|e^{-u'(x)}-P_xP_{X_\varepsilon}\{\mathcal{R}\cap\Gamma=\emptyset,e^{-A'}\}|\\&\quad+|P_xP_{X_\varepsilon}\{\mathcal{R}\cap\Gamma=\emptyset,e^{-A'}\}-e^{-u_\varepsilon(x)}|\le I_\varepsilon+J_\varepsilon,\end{aligned}$$

where

$$I_\varepsilon=P_xP_{X^\varepsilon}\{\mathcal{R}\cap\Gamma=\emptyset,e^{-A'}(1-e^{-A''})\}\le P_xP_{X^\varepsilon}(1-e^{-A''})=1-e^{-V_{D^\varepsilon}(u'')(x)},$$

$$J_\varepsilon=P_xP_{X^\varepsilon}\{\mathcal{R}\cap\Gamma\neq\emptyset,e^{-A'}\}\le1-P_xe^{-\langle w,X^\varepsilon\rangle}\le P_x\langle w,X^\varepsilon\rangle.$$

By (3.2) and 4.4.D, $\lim V_{D_\varepsilon}(u'')=Q_B(u'')=0$. Therefore $I_\varepsilon\to0$. By (1.12), $P_x\langle w,X^\varepsilon\rangle=\Pi_xw(\xi_{\tau_\varepsilon})$, where τ_ε is the first exit time from D_ε. By (5.2), $w=0$ on O, and therefore $\Pi_xw(\xi_{\tau_\varepsilon})\to0$. Therefore $J_\varepsilon\to0$. Hence, $Q_B(u)=\lim u_\varepsilon=u'$.

Formula (5.5) follows from 4.4.C and 3.3.C. □

Lemma 5.1. *Every moderate compact set $B\subset\Gamma$ for a (Γ,ν)-solution u is $\mathcal{R}$-polar.*

Proof. Put $w(x)=-\log P_x\{\mathcal{R}\cap B=\emptyset\}$. By Theorem 3.1, $w_B=Q_B(w)$ is the maximal element of $\mathcal{U}$ subject to the conditions $w_B\le w$ and $w_B=0$ on $\partial E\setminus B$. Since these conditions hold for w, we have $w=w_B$. Since $u\ge w$, $u_B=Q_Bu\ge w_B=w$ by 3.1.A. Since u_B is a moderate solution, w is also moderate and

$$h=w+\mathcal{E}(w) \tag{5.6}$$

is the minimal L-harmonic majorant of w. Since trace$(2h) = 2$ trace(h), it follows from 3.4.B that

$$v + \mathcal{E}(v) = 2h \tag{5.7}$$

for some $v \geq 0$. By 4.4.B, $v \leq 2w$, and therefore $v = 0$ on $\partial E \setminus B$. But w is the maximal solution with this property, and therefore $v \leq w$, which implies $2h \leq h$ and $h = 0$. Hence $w = 0$ and B is $\mathcal{R}$-polar. □

5.3. Normal pairs. We denote by $Ex(\nu)$ the set of all explosion points of a measure ν. If $B \cap Ex(\nu) = \emptyset$ and B is compact, then $\nu(B) < \infty$. A measure ν concentrated on an open set O is a Radon measure on O if and only if $O \cap Ex(\nu) = \emptyset$.

Note that a (Γ, ν)-solution is at the same time a (Γ_0, ν)-solution if $\Gamma_0 \subset \Gamma$ and $\Lambda = \Gamma \setminus \Gamma_0$ satisfies condition (1.19). Indeed, (1.19) implies that $\{\mathcal{R} \cap \Gamma = \emptyset\} = \{\mathcal{R} \cap \Gamma_0 = \emptyset\}$ P_x-a.s. for all $x \in E$. We call (Γ, ν) a *normal pair* if condition (1.19) holds only for $\Lambda = \emptyset$.

Lemma 5.2. *Every (Γ, ν)-solution u is also a (Γ_0, ν)-solution with a normal pair (Γ_0, ν).*

Proof. The class $\mathcal{L}(\Gamma, \nu)$ of all sets Λ subject to condition (1.19) is closed under union.[9] Denote by Λ_0 the union of all $\Lambda \in \mathcal{L}(\Gamma, \nu)$, and put $\Gamma_0 = \Gamma \setminus \Lambda_0$. Since $\Lambda_0 \in \mathcal{L}(\Gamma, \nu)$, u is a (Γ_0, ν)-solution. If $\Lambda \in \mathcal{L}(\Gamma_0, \nu)$, then $\Lambda \cup \Lambda_0 \in \mathcal{L}(\Gamma, \nu)$, and therefore $\Lambda \subset \Lambda_0$. On the other hand, $\Lambda \subset \Gamma_0 = \Gamma \setminus \Lambda_0$. Hence, $\Lambda = \emptyset$. □

5.4.

Theorem 5.2. *Let u be a (Γ, ν)-solution. If (Γ, ν) is a normal pair, then the trace of u is equal to (Γ, ν).*

Proof. Denote the trace of u by (Γ_0, ν_0). By the definition of the trace (see Section 3.5), $O_0 = \partial E \setminus \Gamma_0$ is the maximal moderate open set for u. Hence, $O_0 \supset O = \partial E \setminus \Gamma$.

For every compact set $B \subset O$, the trace of $Q_B(u)$ is equal to the restriction of ν_0 to B. By Theorem 5.1, it is equal to the restriction of ν to B. Therefore $\nu = \nu_0$ on O. Since ν is concentrated on O, we have $\nu \leq \nu_0$, and therefore $Ex(\nu) \subset Ex(\nu_0) \subset \Gamma_0 \subset \Gamma$. The set $\Lambda = \Gamma \setminus \Gamma_0$ is $\mathcal{R}$-polar. Indeed, every compact $B \subset \Lambda$ is moderate for u (because $B \subset O_0$), and it is $\mathcal{R}$-polar by Lemma 5.1. Thus, Λ is $\mathcal{R}$-polar, $\Lambda \cap Ex(\nu) = \emptyset$ and $\Gamma \setminus \Lambda = \Gamma_0$ is closed. By the definition of a normal pair, this implies $\Lambda = \emptyset$. □

5.5.

Theorem 5.3. *The trace (Γ, ν) of every solution u is a normal pair.*

First we prove two lemmas.

Lemma 5.3. *If $D_1, D_2, \tilde{D}$ are open subsets of E and if*

$$D_1 \cap \tilde{D} = D_2 \cap \tilde{D}, \tag{5.8}$$

then

$$\{\mathcal{R} \subset \tilde{D}\} \subset \{X_{\tilde{D}} = 0\} \subset \{X_{D_1} = X_{D_2}\} \quad P_x\text{-a.s.} \tag{5.9}$$

[9] The union of an arbitrary family of $\mathcal{R}$-polar sets relatively open in Γ is $\mathcal{R}$-polar.

Proof. Let B_n be compact sets such that $B_n \uparrow \tilde{D}$. By [2, Lemma 2.1], $\{\mathcal{R} \subset B_n\} \subset \{X_{\tilde{D}} = 0\}$ a.s., which implies the first part of (5.9).

The set $Q = \{X_{\tilde{D}} = 0, X_{D_1} \neq X_{D_2}\}$ belongs to $\mathcal{F}_{\supset U}$, where $U = \tilde{D} \cap D_1 = \tilde{D} \cap D_2$. By (1.11), $P_x(Q) = P_x P_{X_U}(Q)$, and, by [4, II.4.6], the second part of (5.9) will be proved if we show that $P_\eta(Q) = 0$ for every measure $\eta \in \mathcal{M}(U^c)$. Let η' be the restriction of η to $\tilde{D}^c$. Note that $X_{\tilde{D}} = \eta'$ $P_{\eta'}$-a.s. If $\eta' \neq 0$, then $P_\eta\{X_{\tilde{D}} = 0\} \leq P_{\eta'}\{X_{\tilde{D}} = 0\} = 0$ and therefore $P_\eta(Q) = 0$. If $\eta' = 0$, then η is concentrated on the complement of $D_1 \cup D_2$. Hence, $X_{D_1} = X_{D_2} = \eta$ P_η-a.s., and again $\mathcal{P}_\eta(Q) = 0$. □

Lemma 5.4. *If (Γ, ν) is the trace of u and if Γ is $\mathcal{R}$-polar and ν is finite, then u is moderate.*

Remark. We know (see Section 3.5) that if u is moderate, then Γ is empty.

Proof. Put $B_n = \{x \in \partial E : d(x, \Gamma) \geq 1/n\}$. We have $B_n \uparrow O = \partial E \setminus \Gamma$. By (3.5) and (3.6),

$$u(x) = -\log P_x \exp\{-Z\},$$
$$u_n(x) = Q_{B_n}(u)(x) = -\log P_x \exp\{-Z_{B_n}\}.$$

The trace of u_n is the restriction of ν to B_n, and therefore

$$u_n(x) + \mathcal{E}(u_n)(x) = \int_{B_n} k(x,y)\nu(dy) < \infty.$$

Therefore, for all x and n,

$$u_n(x) \leq h(x) = \int_O k(x,y)\nu(dy).$$

By 3.3.A, there exists $\tilde{Z}$ such that $Z_{B_n} \uparrow \tilde{Z}$ P_x-a.s. for all x. We have

$$-\log P_x e^{-\tilde{Z}} = \lim u_n(x) \leq h(x).$$

To prove that u is moderate, it is sufficient to show that, for every x,

$$\tilde{Z} = Z \quad P_x\text{-a.s.} \tag{5.10}$$

We apply Lemma 5.3 to $D_1 = D(B_n, \varepsilon), D_2 = D(\partial E, \varepsilon), \tilde{D} = D(\Gamma, \delta)$. Condition (5.8) holds if $\varepsilon < \delta - 1/n$. We conclude from (5.9) that

$$P_x\{\mathcal{R} \subset D(\Gamma,\delta), \exp[-\langle u, X_{D(B_n,\varepsilon)}\rangle]\} = P_x\{\mathcal{R} \subset D(\Gamma,\delta), \exp[-\langle u, X_{D(\partial E,\varepsilon)}\rangle]\}.$$

By passing to the limit, first as $\varepsilon \to 0$ using 3.3.C, then as $n \to \infty$ and, finally, as $\delta \to 0$, we get

$$P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-\tilde{Z}}\} = P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-Z}\}.$$

Since Γ is $\mathcal{R}$-polar and since $P_x\{\tilde{Z} \leq Z\} = 1$ by 3.3.A, we get (5.10). □

Proof of Theorem 5.3. Let $\Lambda \in \mathcal{L}(\Gamma, \nu)$ and let $\Gamma_0 = \Gamma \setminus \Lambda$. The theorem will be proved if we show that $v = Q_{B_0}(u)$ is moderate for every closed subset B_0 of $O_0 = \partial E \setminus \Gamma_0$. Indeed, this implies $O_0 \subset O$, and therefore $\Gamma_0 \supset \Gamma$, $\Lambda = \emptyset$.

Let (Γ_1, ν_1) be the trace of v. By Lemma 5.4, it is sufficient to to prove that Γ_1 is $\mathcal{R}$-polar and ν_1 is finite.

By 3.1.B, $v \leq u$, and by 3.1.A, all moderate sets for u are also moderate for v. Hence $\Gamma_1 \subset \Gamma$. By 3.2.B, $Q_B[Q_{B_0}](u) = 0$ for all $B \subset \partial E \setminus B_0$. Hence, $\partial E \setminus B_0$ is moderate for v and it is contained in $O_1 = \partial E \setminus \Gamma_1$. We conclude that $\Gamma_1 \subset B_0 \cap \Gamma$ which is a subset of the $\mathcal{R}$-polar set Λ.

Note that $B_0 \subset O \cup \Lambda$ does not contain explosion points of ν and therefore $\nu(B_0) < \infty$. Since $Q_B(v) = 0$ for $B \cap B_0 = \emptyset$, the measure ν_1 vanishes on $\partial E \setminus B_0$. Since $v \le u$, $\nu_1 \le \nu$ on $O \subset O_1$. We have

$$\nu_1(O_1) = \nu_1(O_1 \cap B_0) = \nu_1[(O_1 \setminus O) \cap B_0] + \nu_1(O \cap B_0) \le \nu(O \cap B_0) < \infty$$

because $\Gamma \cap B_0$ is polar, and therefore $\nu_1(\Gamma \cap B_0) = 0$ by Theorem 1.1. □

6. Maximal property of (Γ, ν)-solutions

6.1. In this section we consider an arbitrary solution u with the trace (Γ, ν). By 3.5.A, ν does not charge $\mathcal{R}$-polar sets, and, by Theorems 4.3 and 4.2, it is the spectral measure of a CLA functional A^ν. Our objective is to prove:

Theorem 6.1. *We have*

$$u(x) \le -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-A^\nu}\}. \tag{6.1}$$

This follows easily from a kind of a mean value theorem which is of independent interest:

Theorem 6.2. *Let u be a solution with the trace (ν, Γ). For every regular open subset D of E such that $\partial D \cap \Gamma = \emptyset$, there exists a random variable Z^D with the properties*

$$Z^D \le A^\nu; \tag{6.2}$$

$$u(x) = -\log P_x e^{-Z^D - \langle u, X_D \rangle} \quad \textit{in } D. \tag{6.3}$$

To prove Theorem 6.1, we apply Theorem 6.2 to domains $D_n = D(\Gamma, 1/n)$. It follows from (6.2) and (6.3) that

$$u(x) \le -\log P_x e^{-A^\nu} e^{-\langle u, X_{D_n} \rangle} \tag{6.4}$$

for every n. Since $\langle u, X_{D_n} \rangle \to 0$ on the set $\{\mathcal{R} \cap \Gamma = \emptyset\}$, we have $\liminf e^{-\langle u, X_{D_n} \rangle} \ge 1_{\mathcal{R} \cap \Gamma = \emptyset}$, and Fatou's lemma implies that

$$\liminf P_x e^{-A^\nu} e^{-\langle u, X_{D_\varepsilon} \rangle} \ge P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-A^\nu}\}. \tag{6.5}$$

Equations (6.4) and (6.5) imply (6.1).

6.2.

Proof of Theorem 6.2. 1°. Put $B = \partial D \cap \partial E$, $D_n = D(B, 1/n)$ and $\tilde{D}_n = D \cap D_n$. For every set A, put $u_A = u 1_A$.

By the mean value property 2.1.D,

$$u(x) = -\log P_x e^{-\langle u, X_{\tilde{D}_n} \rangle} \quad \text{in } \tilde{D}_n. \tag{6.6}$$

For every $x \in \tilde{D}_n$ the measure $X_{\tilde{D}_n}$ is concentrated, P_x-a.s., on $C \cup D$, where $C = \partial D \cap E$, and therefore

$$\langle u, X_{\tilde{D}_n} \rangle = \langle u_C, X_{\tilde{D}_n} \rangle + \langle u_D, X_{\tilde{D}_n} \rangle \quad P_x\text{-a.s.}$$

Theorem 6.2 will follow from (6.6) if we prove that, for every $x \in D$,

$$\lim \langle u_C, X_{\tilde{D}_n} \rangle = \langle u, X_D \rangle \quad P_x\text{-a.s.} \tag{6.7}$$

and

$$\langle u_D, X_{\tilde{D}_n}\rangle \to Z^D \quad P_x\text{-a.s.,} \tag{6.8}$$

where Z^D satisfies (6.2).

2°. Note that $\langle u_C, X_{\tilde{D}_n}\rangle = \langle u, \tilde{X}_n\rangle$ and that, for all $x \in D$, $\langle u, X_D\rangle = \langle u, \tilde{X}\rangle$ P_x- a.s., where $\tilde{X}_n$ and $\tilde{X}$ are the restrictions of $X_{\tilde{D}_n}$ and X_D to C. Therefore we get (6.7) if we prove that $\tilde{X}_n \uparrow \tilde{X}$ P_x-a.s.

By 2.1.E, $\tilde{X}_n \le \tilde{X}_{n+1} \le \tilde{X}$ a.s., and therefore

$$\tilde{X}_n \uparrow X^* \le \tilde{X} \quad \text{a.s.} \tag{6.9}$$

By (1.10) and (1.12),

$$P_x\langle 1, \tilde{X}_n\rangle = \Pi_x\{\xi_{\tilde{\tau}_n} \in C\},$$
$$P_x\langle 1, \tilde{X}\rangle = \Pi_x\{\xi_{\tau} \in C\},$$

where $\tilde{\tau}_n, \tau$ are the first exit moments from $\tilde{D}_n, D$. If $x \in D$, then $\{\xi_{\tilde{\tau}_n} \in C\} \uparrow \{\xi_\tau \in C\}$ Π_x-a.s., and (6.9) implies that $X^* = \tilde{X}$ P_x-a.s.

3°. Denote by $\bar{X}_n$ the restriction of $X_{\tilde{D}_n}$ to D. Note that $\bar{X}_{n+1} = 0$ $P_{\tilde{X}_n}$-a.s. By (1.11) and (1.7),

$$P_x\{e^{-\langle u_D, X_{\tilde{D}_{n+1}}\rangle} | \mathcal{F}_{\subset \tilde{D}_n}\} = P_{X_{\tilde{D}_n}} e^{-\langle u_D, X_{\tilde{D}_{n+1}}\rangle} = P_{\bar{X}_n} e^{-\langle u_D, X_{\tilde{D}_{n+1}}\rangle}$$
$$\ge P_{\bar{X}_n} e^{-\langle u, X_{\tilde{D}_{n+1}}\rangle} = e^{-\langle u, \bar{X}_n\rangle} = e^{-\langle u_D, X_{\tilde{D}_n}\rangle}.$$

Therefore $e^{-\langle u_D, X_{\tilde{D}_n}\rangle}$ is a bounded submartingale relative to P_x (cf. the proof of 3.3.C) and there exists Z^D subject to (6.8).

To get (6.2), we recall that, by Section 3.5, the restriction ν_B of ν to B is the trace of $Q_B(u)$. By 4.4.C, the corresponding functional A^{ν_B} satisfies the condition

$$A^{\nu_B} = \lim \langle u, X_{D_n}\rangle \quad \text{a.s.}$$

Proposition 2.1.E implies that

$$\langle u_D, X_{\tilde{D}_n}\rangle \le \langle u_D, X_{D_n}\rangle \le \langle u, X_{D_n}\rangle.$$

Hence

$$Z^D = \lim \langle u_D, X_{\tilde{D}_n}\rangle \le A^{\nu_B} \le A^{\nu}.$$

□

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